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Coexistence of stable particle and hole solutions for fixed parameter values in a simple reaction diffusion system

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We present a simple autocatalytic reaction-diffusion model for two variables, which shows for fixed parameter values the simultaneous stable coexistence of particle solutions as well of two types of hole solutions. The associated spatially homogeneous system is characterized by the coexistence of one stable fixed point and a stable limit cycle solution. We compare our results to other dissipative systems which have for fixed parameters either stable particle or stable hole solutions including the quintic complex Ginzburg-Landau equation and the envelope equation for optical bistability as well as other reaction-diffusion models.

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The study of reaction-diffusion (RD) systems is a growing sub field of pattern formation in nonequilibrium systems, which has become a well-established field of physics [1]. One of the most interesting recent developments in autocatalytic chemical reactions is the investigation of localized solutions and pulses. This includes, for example, self-replicating spots [2–4] and the interaction of pulses [5,6]. The study of stable localized solutions and their interaction has attracted increasing attention within the framework of several classes of prototype evolution equations in the study of dissipative systems including the quintic complex Ginzburg-Landau equation [7–9], order parameter equations [10,11], and phase equations [12,13]. Much less work has been done on stable hole solutions in these dissipative prototype equations [14,15] (compare also the review in [16]). For optical bistability a transition from stable particle to stable hole solutions has been described [17].

Here we study the question whether it is possible to have for fixed parameters in dissipative evolution equations stable particle as well as stable hole solutions simultaneously and what the underlying mechanisms are to achieve such a situation. Motivated by the very rich behavior well known for simple two variable reaction diffusions systems, we chose a model such that it has for a large parameter range a stable limit cycle coexisting with a stable fixed point. This situation has been established to give rise to stable particle solutions for reaction diffusion systems [6] as well as for the quintic complex Ginzburg-Landau equation [7,8]. We show numerically that in the vicinity of the parameter values for which the speed of wall solutions passes through zero, a class of holes exists stably, which we call 2π holes, embedded in the regime of existence of another class of stable hole solutions (called π holes). For a small range of parameters all three classes of stable localized solutions occur simultaneously: two classes of stable holes as well as stable particle solutions.

The reaction-diffusion model studied has the form

$$\dot{u} = \mu_1 u - \mu_2 v + \beta_r u^3 + \gamma_r u^5 + u_{xx}, \tag{1}$$

$$\dot{v} = \mu_2 u + \mu_3 v + \beta_i u^3 + D v_{xx}, \tag{2}$$

where length and time scales have already been rescaled to reduce the number of parameters in the system to the ones that are independent. The parameters β_r and γ_r in Eq. (1) are taken to have a destabilizing cubic term $(\beta_r > 0)$ and a stabilizing quintic term $(\gamma_r < 0)$ to guarantee stability for large values of u. Without the coupling to v, Eq. (1) would give rise for $\mu_1 < 0$ to five stationary points, three of which are stable. Equations (1) and (2) possess the symmetry $u \rightarrow -u$ and $v \rightarrow -v$ simultaneously, but not separately. The linearized version of Eqs. (1) and (2) has the standard structure for reaction-diffusion systems giving rise to oscillatory motions. We note that the term $\sim u^3$ in Eq. (2) is nonpotential in nature.

It turns out that the phase diagram associated with Eqs. (1) and (2) is in general extremely rich and complex. A detailed discussion will therefore be deferred to a longer paper [19]. Here we focus on the case where the dynamical system associated with Eqs. (1) and (2) has one stable fixed point for $u = v = 0$ and a stable limit cycle. This structural situation, which is sketched in Fig. 1, arises very frequently for reaction diffusion systems and is therefore of direct importance for experimental studies in the field of autocatalytic chemical reactions (compare, for example, Ref. [18].)

Throughout the rest of this paper we chose the parameter values $\beta_r = 3$, $\gamma_r = -2.75$, $\mu_2 = 1.5$, $\mu_1 = 0$, $\mu_3 = -0.2$, and $D = 1$. We then discuss the phenomena that arise as a function of β_i. In this paper we use periodic boundary conditions. We have also checked other boundary conditions (von Neumann) to verify that there is no qualitative change for the phenomena observed upon changing the boundary conditions. Here we address the question which types of stable localized solutions occur in addition to the stable spatially homogeneous solutions $u = v = 0$ and the stable limit cycle solution. In particular we demonstrate that one can have—*for fixed values of all the parameters in the two equations*—stable particles and

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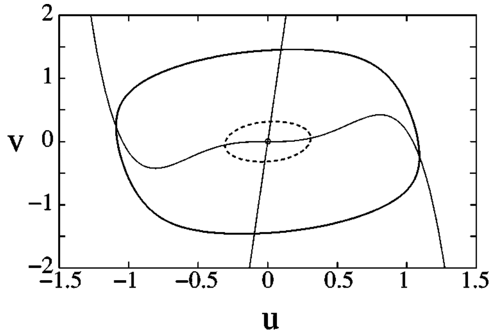

FIG. 1. This plot shows the generic behavior of the dynamical system (without spatial degrees of freedom) for the regime of parameter space considered in this note. The null klines of the ODE's are shown as solid black lines. The origin of the u-v plane corresponds to a stable fixed point. The thick solid black line is a stable limit cycle, while the unstable limit cycle is shown as a dashed line. $\beta_i=1.600$.

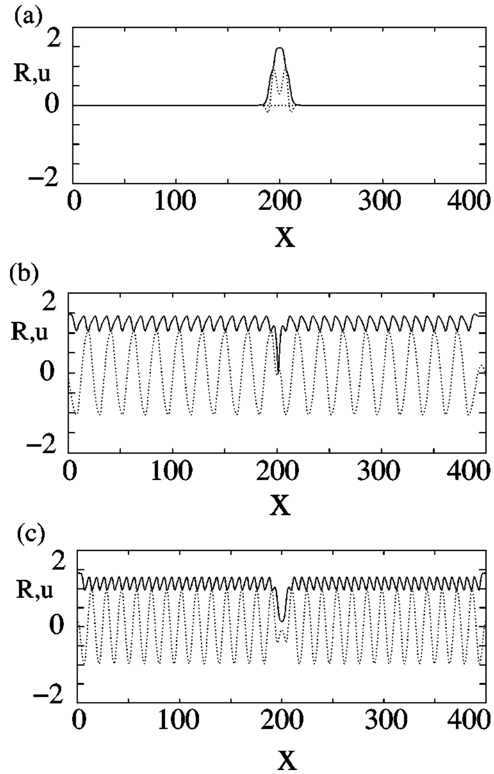

FIG. 3. A snapshot is shown for a particle, a π hole, and a 2π hole as a function of space. The quantities plotted are the modulus $R(x,t)=\sqrt{u^2+v^2}$ and the variable $u(x,t)$. For π holes the modulus reaches zero while this is not the case for 2π holes. The value of β_i is chosen such that all three localized objects stably exist for the same parameter value: $\beta_i=1.600$.

two types of stable holes. In addition we would like to stress that the stable particle solutions and one class of stable hole solutions can appear simultaneously over a fairly large range of values of the parameter β_i. The resulting phase diagram is plotted in Fig. 2. Inspecting Fig. 2 it emerges that stable particles as well as two classes of stable hole solutions occur for positive as well as for negative values of β_i. While the structure and the sequence of localized solutions is similar, there is, however, no $\beta_i \to -\beta_i$ symmetry, which can also be seen immediately by inspecting the parameter values given in the caption of Fig. 2 marking the limits of the occurrence of the various types of solution. Next we characterize the nature of these three classes of stable localized solutions.

Figure 3(a) shows a snapshot of the modulus R, $R=\sqrt{u^2+v^2}$, and the variable u as a function of space. The particle solutions act as a sink of waves traveling towards the particle for negative values of β_i and as a source for positive values of β_i. Their maximum amplitude is comparable to the maximum amplitude of the spatially homogeneous limit cycle solution. In addition, the modulus of the particlelike solutions carries out a periodic breathing motion of the maximum amplitude—because the limit cycle is not a circle—sending out traveling waves, which become particularly noticeable in the local wave vector. This phenomenon as well as an analytic approximation scheme to capture all essential ingredients of these breathing particles will be described in [20], generalizing the technique applied previously for fixed shape and breathing localized solutions of the quintic complex Ginzburg-Landau equation [21,22]. As $|\beta_i|$ becomes smaller, the width of the particle solution increases, while it decreases when $|\beta_i|$ becomes larger. When the boundaries for the stable existence of particles are crossed, the particles disappear by a filling in for sufficiently small values of $|\beta_i|$, while they collapse for $|\beta_i|$ sufficiently large. Furthermore, we note that one obtains a period doubling leading to an additional lateral breathing motion for positive values of β_i as one approaches the lower boundary of their stable existence, denoted by β_{p+1} in Fig. 2. This is the analog of the transition to the periodically breathing localized states in the complex quintic Ginzburg-Landau equation.

Figure 3(b) shows a snapshot of a π hole. The reason for this notation becomes clear from an inspection of the phase portraits shown in Figs. 4(b) and 4(e) in which the trajectory goes through the center of the limit cycle. This also demonstrates that π holes reach zero modulus, $R=0$. For large dis-

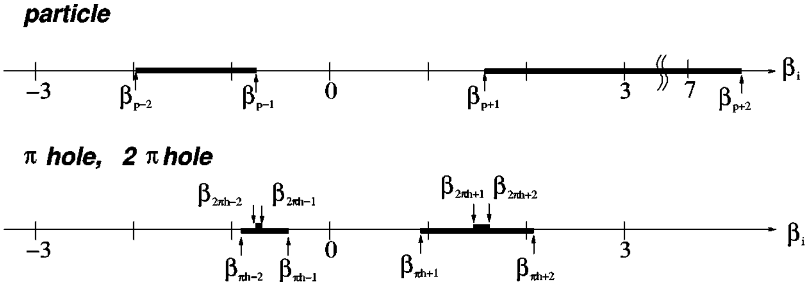

FIG. 2. The phase diagram for the stable existence of particles, π holes, and 2π holes is shown as a function of β_i. The upper and lower boundaries for particles, π and 2π holes for positive and negative values of β_i are given by $\beta_{p-1}=-0.775$, $\beta_{p-2}=-1.988$, $\beta_{p+1}=1.582$, $\beta_{p+2}=7.655$, $\beta_{\pi h-1}=-0.432$, $\beta_{\pi h-2}=-0.877$, $\beta_{\pi h+1}=0.941$, $\beta_{\pi h+2}=2.165$, $\beta_{2\pi h-1}=-0.735$, $\beta_{2\pi h-2}=-0.779$, $\beta_{2\pi h+1}=1.500$, $\beta_{2\pi h+2}=1.664$

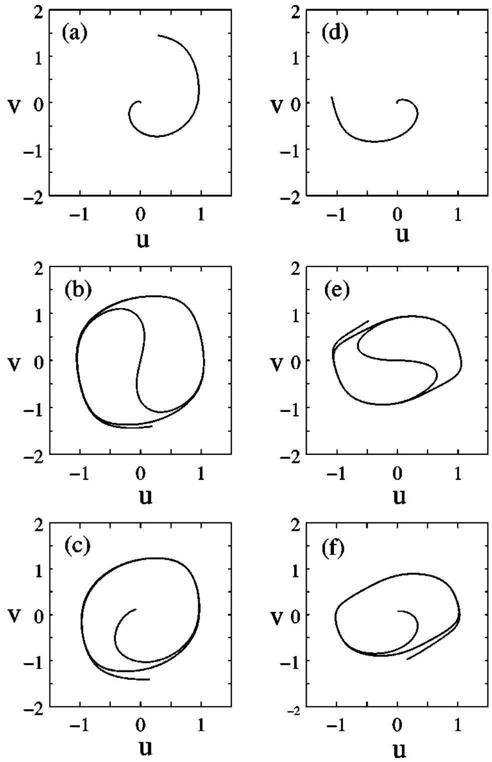

FIG. 4. Phase portraits in the u-v plane are shown for positive and negative values of β_i. For the left column $\beta_i = 1.600$ and for the right column $\beta_i = -0.777$. (a) and (d) show stable particle solutions, (b) and (e) stable π holes, and (c) and (f) stable 2π holes. The curves in the left column are structurally mirror images of the curves in the right column. All curves are rotating counterclockwise. This observation is closely related to the fact that for positive values of β_i particles act as sources of traveling waves, while 2π and π holes act as sinks. For negative values of β_i the opposite situation prevails: particles correspond to sinks, while 2π and π holes act as sources of traveling waves.

tance from the hole, R reaches asymptotically the value of the spatially homogeneous limit cycle solution. In Fig. 5 we show an $x-t$ plot of a π hole from which the phase jump of π at the core is clearly visible. Inspecting Fig. 2, we see that there is a considerable range of parameter values of β_i, both for positive and negative values, over which stable particles as well as stable π holes exist simultaneously. For $\beta_i > 0$ the stable hole solutions acts as sinks—a fact also brought out by Fig. 5—while for $\beta_i < 0$ they act as sources. Correspondingly a source (or a sink) appears elsewhere in the system. For $|\beta_i|$ sufficiently small, the π holes disappear by collapsing while for $|\beta_i|$ sufficiently large they are filling in. We also note that the width of the π holes is approximately constant. For π holes the wavelength is larger for smaller $|\beta_i|$, while it becomes smaller for larger $|\beta_i|$.

Figure 3(c) shows a snapshot of a 2π hole. Its detailed nature with respect to the phase space is brought out in the phase portraits shown in Figs. 4(c) and 4(f). Note that the modulus of the 2π holes stays finite near their center and does not touch 0 in contrast to the case of the π holes. Far away from the center their R value is comparable to that of the spatially homogeneous limit cycle solution. For negative

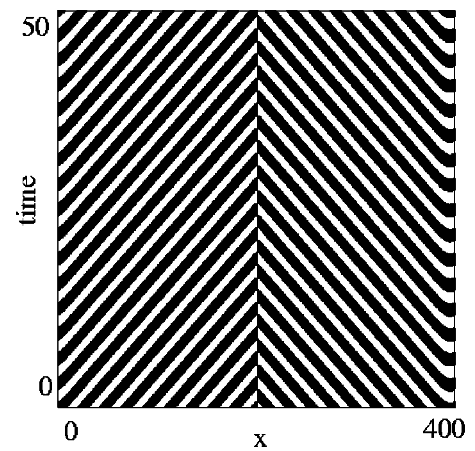

FIG. 5. We show an x-t plot for a π hole and $\beta_i = 1.600$. The quantity shown is $u(x,t)$; $u > 0$: white, $u < 0$: black. From the plot it is clear that the π holes act as a sink and that the phase jump at annihilation is π. The rounding off in the x-t plot near $x = 0$ and $x = 400$ is caused by the periodic boundary conditions used.

values of β_i, 2π holes act as sources while for positive values of β_i they act as sinks. 2π holes exist stably only over a narrow parameter range, which is included in the range of existence of stable π holes. It appears most important to stress that the existence of stable 2π holes is connected with the region of parameters for which the speed of wall solutions passes through zero. In addition, their range of stable existence overlaps with that of stable particle solutions. Thus we can get for a range of parameters for a fixed value of all the parameters in the equation the simultaneous stable existence of particles, π holes, and 2π holes. For sufficiently large values of $|\beta_i|$ the 2π holes vanish by filling in while for sufficiently small values of $|\beta_i|$ the 2π holes vanish by making a transition to a π hole. We have tested the stability of both, 2π holes and π holes against noisy perturbations. As expected from their limited range of existence and from their transitions to π holes for sufficiently small values of $|\beta_i|$, the 2π holes are more sensitive to noisy perturbations: one needs a noise amplitude that is about one order of magnitude higher to annihilate a π hole. The latter noise amplitude is comparable to that necessary to destroy a breathing localized solution of the complex quintic Ginzburg-Landau equation [9].

The particle solutions discussed resemble most closely the fixed shape localized states for the quintic complex Ginzburg-Landau equation [9]. In this case one also has a stable limit cycle solution and a stable fixed point. There is also an example from reaction diffusion systems for which one has observed laterally breathing localized pulses [6], which occurs under the same conditions: namely, the stable coexistence of a limit cycle and a stationary fixed point. While one has seen a transition from a particle solution to a hole solution as a function of one parameter for the case of optical bistability [17], we are not aware of any report of the simultaneous existence of stable particle and hole solutions for a fixed set of parameters in an underlying evolution equation for a dissipative system, regardless whether this would be an envelope equation [23,24], a phase equation [12], an order parameter equation [11,25], or a reaction-diffusion system [26]. We note that for the latter systems we have in mind a particle or hole in all concentrations; a dip in one concentration and an enhancement in the other are well documented [27] and of completely different origin. Similarly there also appears to exist no report in the literature of the stable and simultaneous existence of two types of holes, such as the π and 2π holes described here, for fixed parameters in a dissipative model.

In conclusion we have shown in this Rapid commnnication that three different types of localized solutions can stably exist simultaneously for a fixed set of parameter values in a simple reaction-diffusion system for two variables, which allows for the coexistence of a stable limit cycle solution and a stable fixed point. Since the RD system studied has properies that are common to large class of models, we soon expect experimental observations of the phenomena predicted here.

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